The International Journal Of Engineering And Science (IJES)
||Volume||2 ||Issue|| 4 ||Pages|| 46-53||2013||
ISSN(e): 2319 – 1813 ISSN(p): 2319 – 1805

On Oscillation And Asymptotic Behaviour Of A Higher Order Functional Difference Equation Of Neutral Type-II

[1,] Radhanath Rath, [2,] Ajit Kumar Bhuyan , [3,] Subhendu Kumar Rath

Abstract

Suffcient conditions are obtained for the oscillation of all the solutions of the neutral functional difference equation

$$\Delta^m(y_n - p_n y_{\tau(n)}) + q_n G(y_{\sigma(n)}) = f_n.$$

Different ranges of $\{p_n\}$ are considered. In particular, we extend the results of [1] Karpuz etal (2009) to the case when G is sublinear.

Key words and phrases. *Oscillatory solution, nonoscillatory solution, asymptotic behaviour, difference equations. 2000 Mathematics Subject Classification. 39A10, 39A12.*

Date Of Submission: 06 March 2013 Date Of Publication: 05,May.2013

I. INTRODUCTION

In this paper, sufficient conditions are obtained so that every solution of

$$\Delta^m(y_n - p_n y_{\tau(n)}) + q_n G(y_{\sigma(n)}) = f_n. \qquad (1.1)$$

oscillates or tends to zero or $\pm\infty$ as $n \to \infty$, where Δ is the forward difference operator given by $\Delta x_n = x_{n+1} - x_n$, p_n, q_n and f_n are infinite sequences of realnumbers with $q_n \geq 0, G \in C(R, R)$.
Further we assume $\{\tau(n)\}$, $\{\sigma(n)\}$ are monotonic increasing and unbounded sequences such that $\tau(n) \leq n$, $\sigma(n) \leq n$ for every n. Different ranges of $\{p_n\}$are considered. The positive integer m can take both odd and even values. Let N_1 be a fixed nonnegative integer. Let $N_0 = \min\{\tau(N1), \sigma(N1)\}$. By a solution of (1.1), we mean a real sequence $\{y_n\}$which is defined for all positive integer $n \geq N_0$ and satisfies (1.1) for $n \geq N_1$. Clearly if the initial condition

$$y_n = a_n \quad \text{for} \quad N_0 \leq n \leq N_1 + m - 1 \qquad (1.2)$$

is given then the equation (1.1) has a unique solution satisfying the given initial condition (1.2).
A solution $\{y_n\}$ of (1.1) is said to be oscillatory if for every positive integer $n_0 > N_1$, there exists $n \geq n_0$ such that $y_n y_{n+1} \leq 0$, otherwise $\{yn\}$ is said to be non-oscillatory. The function G is said to have linear growth (or to be linear) at infinity, if $\lim_{x\to\infty} |G(x)|/x$ is a positive constant. G is super-linear if $\lim_{x\to\infty} |G(x)|/x = \infty$, and G is sub-linear if $\lim_{x\to\infty} |G(x)|/x = 0$.
In the sequel, we shall need the following conditions.
(H0) G is nondecreasing and $xG(x) > 0$ for all real $x \neq 0$.
(H1) $\sum_{n=0}^{\infty} q_n = \infty$
(H2) There exists a bounded sequence $\{F_n\}$ such that $\Delta^m F_n = f_n$ and $\lim_{n\to\infty} F_n = 0$.

Remark 1.1. If the condition $|\sum_{n=n_0}^{\infty} n^{m-1} f_n| < \infty$ is satisfied, then (H2) holds.
Indeed, if we define

$$F_n = \frac{(-1)^m}{(m-1)!} \sum_{j=n}^{\infty} (j - n + m - 1)^{(m-1)} f_j.$$

Then $\Delta^m F_n = f_n$ and $\lim_{n\to\infty} F_n = 0$. Thus (H2) holds.
We assume that p_n satisfies one of the following conditions in this paper.
(A1) $0 \leq p_n \leq b < 1$, (A2) $-1 < -b \leq p_n \leq 0$,

(A3) $-b_2 \le p_n \le -b_1 < -1$, (A4) $1 < b_1 \le p_n \le b_2$,
(A5) $0 \le p_n \le b_2$, (A6) $-b_2 \le p_n \le 0$,
(A7) $1 \le p_n \le b_2$,
where $b, b_1, b2$ are positive real numbers.

In recent years, several papers on oscillation of solutions of neutral delay difference equations have appeared, see [1]–[13] and the references cited there in.
Sufficient conditions for oscillation of

$$\Delta^m (y_n - p_n y_{n-k}) + q_n G(y_{n-l}) = f_n. \tag{1.3}$$

are studied in [10]. In that paper, p_n is confined to (A2) only and G is restricted with a sub-linear condition

$$\left| \int_0^{\pm C} \frac{du}{G(u)} \right| < \infty. \tag{1.4}$$

In [12] the authors find sufficient conditions for the oscillation of solutions of neutral equation

$$\Delta^m (y_n - p_n y_{n-l}) + q_n y_{n-k}^{\alpha} = 0, \tag{1.5}$$

where $\alpha < 1$, is a quotient of odd integers and p_n satisfies (A1) or (A2). Moreover, we observe that the existing papers in the literature do not have much to offer when p_n satisfies (A4), (A6) or (A7). In this direction we find that, the authors in [5] have obtained sufficient conditions for the oscillation of solutions of the equation

$$\Delta^m (y_n - p_n y_{n-k}) + q_n G(y_{n-r}) = 0, \tag{1.6}$$

with (A4) or (A7) and presented some results which are proved to be wrong (see [1, page 72] for the results and counter example).
In a recent publication [1], the authors considered (1.1) and most of the results hold for G satisfying

(H3) $\liminf_{|u| \to \infty} \dfrac{G(u)}{u} \ge \delta > 0$

Due to this assumption these results cannot be applied to (1.1) when $G(x) = x^{\alpha}$ and $\alpha < 1$. To make things more clear, consider the neutral equation (1.1) in the following particular case.

Example 1.2.

$$\Delta^m \left(y_n - \frac{1}{8} y_{n-1}\right) + \frac{1}{2^{9\alpha}} y_{n-3}^{\alpha} = 2^{-3n\alpha}, \tag{1.7}$$

where m is any integer ≥ 1, α is the quotient of any two odd integers. clearly $y_n = 2^{-3n}$ is a solution of (1.7), which tends to zero as $n \to \infty$. If $\alpha < 1$ then in this case, $G(x) = x^{\alpha}$, does not satisfy (H3) and the results in [1] fails to answer the behaviour of solutions of this neutral difference equation. The above example shows that due to the restriction (H3) in [1], a class of neutral difference equations are left out. Hence, we remove this restriction in the present work to accomodate the class of sublinear equations. Moreover, our results hold true for homogeneous equations associted with (1.1).

II. MAIN RESULT

To begin with, we state a lemma, that would be useful for our work. The following lemma which can be easily proved, generalizes [4, Lemma 2.1].

Lemma 2.1. *Let $\{f_n\}$, $\{q_n\}$ and $\{p_n\}$ be sequences of real numbers defined for $n \ge N_0 > 0$ such that*

$$f_n = q_n - p_n q_{\tau(n)}, \quad n \ge N_1 \ge N_0,$$

where $\tau(n) \le n$, is member of a monotonic increasing unbounded sequence. Suppose that p_n satisfies *one of conditions (A2), (A3) or (A5). If* $q_n > 0$ *for* $n \geq N_0$, $\liminf_{n\to\infty} q_n = 0$ *and* $\lim_{n\to\infty} f_n = L$ *exists then* $L = 0$.

Theorem 2.2. *Let* $m \geq 2$. *Suppose that,* p_n *satisfies one of the conditions (A1) or (A2). If (H0), (H1) and (H2) hold, then every solution of (1.1) oscillates or tends to zero as* $n \to \infty$.

Proof. Let $y = y_n$ *be an non-oscillatory solution of (1.1) for* $n \geq N_1$. *Then* $y_n > 0$ *or* $y_n < 0$. *Suppose* $y_n > 0$ *eventually. There exits positive integer* $n_0 \geq N_1 > 0$ *such that* $y_n > 0$, $y_{\tau(n)} > 0$, *and* $y_{\sigma(n)} > 0$ *for* $n \geq n_0$.. *For simplicity of notation, define for* $n \geq n_0$,

$$z_n = y_n - p_n y_{\tau(n)}. \tag{2.1}$$

Set,

$$w_n = z_n - F_n. \tag{2.2}$$

Then using (2.1)–(2.2) in (1.1), we obtain

$$\Delta^m w_n = -q_n G(y_{\sigma(n)}) \leq 0. \tag{2.3}$$

Hence $w_n, \Delta w_n, \ldots, \Delta^{m-1} w_n$ are monotonic and single sign for $n \geq n_1 \geq n_0$.

Then $\lim_{n\to\infty} w_n = \lambda$, where $-\infty \leq \lambda \leq +\infty$. We claim that y_n is bounded. If not then there exists a sub sequence $\{y_{n_k}\}$ such that

$$n_k \to \infty, \quad y_{n_k} \to \infty \quad \text{as} \quad k \to \infty,$$

and

$$y(n_k) = \max\{y_n : n_1 \leq n \leq n_k\}. \tag{2.4}$$

We may choose n_k large enough so that for $\tau(n_k) \geq n_1$, $\sigma(n_k) \geq n_1$. Then from (H2) it follows that, for $0 < \epsilon$, we can find a positive integer n_2 such that, for $k \geq n_2 \geq n_1$ implies $|F_{n_k}| < \gamma$, for some constant $\gamma > 0$.

Consider the first case that p_n satisfies (A1). Hence for $k \geq n_2$, we have

$$w_{n_k} \geq y_{n_k}(1 - b) - \gamma.$$

If we take the limit $k \to \infty$, then we find $\lim_{n\to\infty} w_n = \infty$, because of the monotonic nature of w_n. Hence $w_n > 0, \Delta w_n > 0$ for $n \geq n_2 \geq n_1$. Since $\Delta^m w_n \equiv 0$ and is non-positive, it follows from (2.3) that $\Delta^{m-1} y_n > 0$ for $n \geq n_3 > n_2$. Choose $0 < \epsilon < w_{n_3}$. Then by (H2) we have $|F_n| < \epsilon$ for $n > n_4 > n_3$. Hence for $n > n_4$, $w_n < y_n - F_n < y_n + \varepsilon$, which implies that $0 < w_{n_3} - \epsilon < w_n - \epsilon < y_n$. Setting $u_n = w_n - \epsilon$, $n \geq n_4$, we obtain

$$0 < u_n < y_n, \Delta u_n = \Delta w_n > 0, \ \Delta^{m-1} u_n = \Delta^{m-1} w_n > 0,$$

and

$$\Delta^m u_n = \Delta^m w_n = -q_n G(y_{\sigma(n)}) \leq 0.$$

Summing the above from n_5 to $n - 1$ and using (H0) we obtain

$$\Delta^{m-1}u_n = \Delta^{m-1}u_{n_5} - \sum_{k=n_5}^{n-1} q_k G(y_{\sigma(k)})$$
$$< \Delta^{m-1}u_{n_5} - G(u_{\sigma(n_5)})\sum_{k=n_5}^{n-1} q_k$$

Hence, taking the limit $n \to \infty$ we obtain $\Delta^{m-1}u_n < 0$ by (H1), contradicting $\Delta^{m-1}u_n > 0$ for large n. Hence our claim holds.

Cosider the other case, that is, (A2) is satisfied. Then for $k \geq n_2$, we have

$w_{n_k} \geq y_{n_k} - \gamma$.

Taking the limit $k \to \infty$, we find $\lim_{n\to\infty} w_n = \infty$. Proceeding as in the first case for (A1), we prove that $w_n > 0, \Delta w_n > 0$ and $\Delta^{m-1}w_n > 0$ for $n > n_2$. Since w_n is increasing, we have, for $n \geq n_4 > n_3$

$(1 - b)w_n \leq w_n + p_n w_{\tau(n)}$

$= y_n - F_n - p_n p_{\tau(n)} y_{\tau(\tau(n))} - p_n F_{\tau(n)}$

$\leq y_n - F_n - p_n F_{\tau(n)}$ (2.5)

For $0 < \in < (1 - b)\ w_{n_3}$, there exists $n_4 > n_3$ such that $|F_n| < \in /2,\ for\ n \geq n_4$.

From (2.5) it follows that

$(1 - b)\ w_{n_3} < 1 - b)w_n \leq y_n + \varepsilon / 2\ - p_n \in /2 < y_n + \in$

For $n \geq n_5 > n_4$, because w_n is increasing and $-p_n < 1$. Setting $u_n = (1 - b)w_n - \in$ for $n > n_5$, we obtain

$0 < (1 - b)w_{n3} - \in < u_n < y_n,$

$\Delta u_n = (1 - b)\Delta w_n > 0,\ \ \Delta^{m-1}u_n = (1 - b)\Delta^{m-1}w_n > 0$

and

$$\Delta^m u_n = (1 - b)\Delta^m w_n = -(1 - b)q_n G(y\sigma(n)). \qquad (2.6)$$

Summing (2.6) from n_5 to n − 1 we obtain

$$\Delta^{m-1}u_n < \Delta^{m-1}u_{n_5} - (1-b)\sum_{n_5}^{n-1} q_k G(u_{\sigma(k)})$$
$$< \Delta^{m-1}u_{n_5} - (1-b)G(u_{\sigma(n_5)})\sum_{n_5}^{n-1} q_k,$$

because u_n is increasing. Hence $\Delta^{m-1}u_n < 0$ for large n, due to (H1), a contradiction.

Thus, y_n is bounded. Consequently, whether p_n satisfies (A1) or (A2), w_n is bounded and hence

$$(-1)^{m+k}\Delta^k w_n < 0,\ \ k = 1, 2, \ldots, m - 1, \qquad (2.7)$$

for $n > n_6 > n_1$. If $\liminf_{n\to\infty} y_n = \lambda > 0$ then $y_n > \mu > 0$ for $n > n_7 > n_6$. Hence from (2.3) we get

$$\Delta^{m-1}w_n < \Delta^{m-1}w_{\sigma(n_7)} - G(\mu)\sum_{k=\sigma(n_7)}^{n-1} q_k.$$

Taking limit $n \to \infty$ and using (H1), we obtain $\Delta^{m-1}w_n \to -\infty$ as $n \to \infty$, a contradiction to (2.7). We conclude that $\liminf y_n = 0$. Since w_n is bounded, $\lim_{n\to\infty} w_n$ exists due to (2.7) whether m is odd or even. Inview of (H2) and (2.2), we find that $\lim_{n\to\infty} z_n$ exists. Applying lemma 2.1 to (2.1), we obtain $\lim_{n\to\infty} z_n = 0$. If (A1) holds then for $n > n_1$, $z_n > y_n - by_{\tau(n)}$ implies $y_n \leq z_n + by_{\tau(n)}$. Hence $\limsup_{n\to\infty} y_n < \lim_{n\to\infty} z_n + b \limsup_{n\to\infty} y_{\tau(n)}$, that is, $(1 - b) \limsup_{n\to\infty} yn \leq 0$. Consequently, $\lim_{n\to\infty} y_n = 0$.

If (A2) holds then $y_n \leq z_n\ for\ n \geq n_1$. Hence $\lim_{n\to\infty} y_n = 0$.

If $y_n < 0$, for $n > n_0$ then we set $x_n = -y_n$ in (1.1) for $n \geq n_0$ to obtain

$$\Delta^m(x_n - p_n x_{\tau(n)}) + q_n \widetilde{G}(x_{\sigma(n)}) = \widetilde{f}_n \quad (2.8)$$

where

$$\widetilde{f}_n = -f_n, \widetilde{G}(\upsilon) = -G(-\upsilon). \quad (2.9)$$

Further,

$$\widetilde{F}_n = -F_n, \quad \text{implies} \quad \Delta^m(\widetilde{F}_n) = \widetilde{f}_n. \quad (2.10)$$

Then it can be easily verified that $\widetilde{G}$ satisfy the condition corresponding to the condition (H0) satisfied by G. Also, $\widetilde{F}$ satisfy the condition corresponding

to the condition (H2) satisfied by F. Proceeding as in the proof for the case $y_n > 0$ we obtain $\lim_{n\to\infty} x_n = 0$, that is $\lim_{n\to\infty} y_n = 0$. Thus, the theorem is proved □

As a consequence of Theorem 2.2 we get the following.

Corollary 2.3. *Under the assumptions of theorem 2.2, every nonoscillatory solution of (1.1) tends to zero as $n \to \infty$ and hence, every unbounded solution of (1.1) oscillates.*

Remark 2.4. *Theorem 2.2 remains true if $f_n \equiv 0$.*

Theorem 2.5. *Suppose that (H0) and (H2) hold. Further assume that the following conditions hold.*

(H4) *For $u > 0$ and $v > 0$, there exists $\beta > 0$ such that*

$$G(u) + G(v) \geq \beta G(u + v) \quad \text{and} \quad G(u)G(v) \geq G(uv).$$

(H5) $G(-x) = -G(x)$.

(H6) $\sum_{n_0}^{\infty} q_n^* = \infty$ *where* $q_n^* = \min[q_n, q_{\tau(n)}]$.

(H7) $\sigma(\tau(n)) = \tau(\sigma(n))$

If p_n satisfies (A6) then every solution of (1.1) oscillates or tends to zero as $n \to \infty$.

Proof. Proceeding as in the previous proof, we show $\lim_{n\to\infty} w_n = l$ and $-\infty \leq l < 0$ is not possible. We may observe that $\lim z_n = \lim w_n$ and $y_n \leq z_n$ due to (H2) and (A6) respectively. Hence $l = 0$ implies $\lim_{n\to\infty} y_n = 0$. Assume, if possible, that $0 < l \leq \infty$. Hence $z_n > \lambda > 0$ and $\Delta^{m-1} w_n > 0$ for $n \geq n_2 > n_1$. Then using (H0), (H4)–(H7), we deduce from (2.3) that

$$0 = \Delta^m w_n + q_n G(y_{\sigma(n)}) + G(-p_{\sigma(n)})[\Delta^m w_{\tau(n)} + q_{\tau(n)} G(y_{\tau(\sigma(n))})]$$

$$\geq \Delta^m w_n + G(b_2)\Delta^m w_{\tau(n)} + \beta q_n^* G(z_{\sigma(n)})$$

$$\geq \Delta^m w_n + G(b_2)\Delta^m w_{\tau(n)} + \beta G(\lambda)\, q_n^*.$$

Summing the above inequality and using (H6) we get

$$\Delta^{m-1} w_n + G(b_2)\Delta^{m-1} w_{\tau(n)} < 0$$

for large n, a contradiction. If $y_n < 0$, eventually for large n, then we may proceed with $x_n = -y_n$ as in the proof of the Theorem 2.2 and note that, x_n is a positive solution of (2.8) with (2.9) and (2.10). Further, we note that, (H5) implies $G = \widetilde{G}$. Then proceeding as above, in the proof for the case $y_n > 0$, we prove that $\lim_{n\to\infty} y_n = 0$ and complete the proof of the theorem.

Remark 2.6. (H6) *implies* (H1) *but not conversely.*

Remark 2.7. The prototype of the function G satisfying (H0), (H4), and (H5) is $G(u) = (\beta + |u|^{\mu})|u|^{\lambda} sgn(u)$, where $\lambda > 0, \mu > 0, \lambda + \mu \geq 1, \beta \geq 1$. For verification we may take help of the well known inequality (see [14, p. 292])

$$u^p + v^p \geq \begin{cases} (u+v)^p, & 0 \leq p < 1, \\ 2^{1-p}(u+v)^p, & p \geq 1. \end{cases}$$

Definition 2.8. *For any positive integer* $n \geq n_0$, *define*
$\tau^{-1}(n) = \{m : m \text{ is an integer } \geq n \text{ and } \tau(m) = n\}$.

Theorem 2.9. *Suppose that m is odd and (H0), (H1) hold. If* p_n *satisfies (A7) then every nonoscillatory solution of*

$$\Delta^m(y_n - p_n y_{\tau(n)}) + q_n G(y_{\sigma(n)}) = 0 \tag{2.11}$$

tends to $\pm\infty$.

Proceeding as in the proof of theorem 2.2, we deduce from (2.3) that

$$\lim_{n\to\infty} \Delta^{m-1} z_n = \lambda, \quad -\infty \leq \lambda < \infty.$$

Suppose that $-\infty < \lambda < \infty$. Then (2.7) holds and we can show $\liminf_{n\to\infty} y_n = 0$, as in the proof of theorem 2.2. Then there exists a subsequence y_{n_k} such that $n_k \to \infty$ and $y_{n_k} \to 0$ as $k \to \infty$. As $z_{n_k} < y_{n_k}$, we have $\limsup z_{n_k} \leq 0$. Hence $\lambda \leq 0$. Similarly

$$z_{\tau^{-1}(n_k)} = y_{\tau^{-1}(n_k)} + p_{\tau^{-1}(n_k)} y_{n_k} > -b_2 y_{n_k} \tag{2.12}$$

This implies $\liminf_{n\to\infty} z_{\tau^{-1}(n_k)} \geq 0$. Hence $\lambda \geq 0$. Thus $\lambda = 0$. Consequently, $(-1)^{m+k}\Delta^k z_n < 0$, $k = 0, 1, \ldots, m-1$, for large n, and $\lim_{n\to\infty} \Delta^k z_n = 0, k = 0, 1, \ldots, m-1$. Since m is odd, $\Delta z_n < 0$ for large n. Hence $z_n > 0$ for $n \geq n_2$. This implies $y_n > y_{\tau(n)}$ which further implies $\liminf_{n\to\infty} y_n > 0$, a contradiction. Thus, $\lambda = -\infty$. From (2.12), we have $y_{n_k} > -\frac{1}{b_2} Z_{\tau^{-1}_{n_k}}$. This implies that $\lim_{n\to\infty} y_n = \infty$. Thus the proof of the theorem is complete.

Remark 2.10. Al l the above results hold when G is linear, super-linear, or sub-linear. Next, we give few examples to establish the significance of our results.

Example 2.11. Consider the neutral equation

$$\Delta^m\left(y_n - \frac{1}{2} y_{n-1}\right) + n^{-1} y_{n-2}^{\alpha} = n^{-1} 2^{\alpha(2-n)}, \tag{2.13}$$

Where $m \geq 2, \alpha$ is a positive rational, being the quotient of two odd integers. In this case, $p_n = \frac{1}{2}$ satisfies (A1). Further, $q_n = n^{-1}, G(x) = x^{\alpha}$ and $f_n = n^{-1} 2^{\alpha(2-n)}$. It is clear that $\sum_{n=n_0}^{\infty} n^{m-1} f_n < \infty$. Hence by Remark 1.1,it follows that

$$F_n = \frac{(-1)^m}{(m-1)!} \sum_{j=n}^{\infty} (j - n + m - 1)^{(m-1)} j^{-1} 2^{\alpha(2-j)}.$$

Obviously, F_n satisfies (H2). Clearly, the equation (2.13) satisfies all the conditions of Theorem 2.2. Hence every non-oscillatory solution tends to zero as $n \to \infty$. In particular $y_n = 2^{-n}$ is a non-oscillator y solution of (2.13),which tends to zero as $n \to \infty$. However, if $\alpha < 1$, then the results of [1] cannot be applied to the neutral difference equation (2.13), because (H3) is not satisfied. Again if $\alpha \geq 1$ then results of [10,12] fail, due to the assumption of a sublinear condition on G, in these papers.

Example 2.12. Consider the neutral equation

$$\Delta^m\left(y_n+\frac{1}{2}y_{n-1}\right)+n^{-1}y_{n-2}^{\alpha}=(-1)^m\,2^{-n-m+1}+n^{-1}2^{\alpha(2-n)}, \qquad (2.14)$$

where $m \geq 2$, α is a positive rational, which is the quotient of two odd integers. Here, $p_n=-\frac{1}{2}$ satisfies (A2). Also, $q_n = n^{-1}$, $G(x) = x^{\alpha}$ and $f_n = (-1)^m 2^{-n-m+1} + n^{-1}2^{\alpha(2-n)}$. Easily, we can verify that, $\sum_{n=n_0}^{\infty} n^{m-1} f_n < \infty$ and the equation (2.14) satisfies all the conditions of Theorem 2.2 for (A2). Hence $y_n = 2^{-n}$ is a positive solution of (2.14), which tends to zero as $n \to \infty$. However,

if $\alpha < 1$, then the results of [1] cannot be applied to this equation, because (H3) is not satisfied. Again if $\alpha \geq 1$ then results of [10,12] fail as G is sublinear there.

Example 2.13. Consider the neutral equation

$$\Delta^m(y_n+8y_{n-2})+\frac{1}{n}y_{n-1}^{\alpha}=(-1)^m\left(1+2^9\right)\left(1-\frac{1}{2^3}\right)^m 2^{-3n}+\frac{2^{\alpha(3-3n)}}{n}, \qquad (2.15)$$

where $m \geq 2$, α is a positive rational, which is the quotient of two odd integers. In this case, $p_n = -8$, satisfies (A3). Again, $q_n = n^{-1}$, $q_n^* = \min[q_n, q_{\tau(n)}] = \frac{1}{n}$.

Further, we find that $G(x) = x^{\alpha}$ and $f_n = (-1)^m(1+2^9)(1-\frac{1}{2^3})^m 2^{-3n} + \frac{2^{\alpha(3-3n)}}{n}$.

The above neutral equation satisfies all the conditions of Theorem 2.5. Hence $y_n = 2^{-3n}$ is a non-oscillatory solution of (2.15), which tends to zero as $n \to \infty$. However, if $\alpha < 1$, then the results of [1] cannot be applied to this equation, because (H3) is not satisfied. Again if $\alpha \geq 1$ then results of [10,12] fail as G is sublinear there.

Before we close we give our final comments which may be helpful for further research.

III. FINAL COMMENTS

In this article, our Theorem 2.2 deals with the range (A1) or (A2) for p_n. Can we get some result like Theorem 2.2 with (A1) and (A2) replaced bythe conditions $0 \leq p_n \leq 1$ and $-1 \leq p_n \leq 0$ respectively. Further, when p_n lies in (A6), wehavedoneTheorem 2.5 under (H6) which is stronger than (H1).Someone may attempt with a changed technique and method to do a result similar to Theorem 2.5 under the weaker condition (H1). Our result Theorem 2.9 should be improved. An attempt may be made to do it for the non-homogeneous neutral equation (1.1). Most significantly, we find majority of the results including this paper, assume q_n to have fixed sign. Results with q_n changing sign (see [7] for first order neutral difference equation) are rare. One may extend and improve the existing results on q_n changing sign to higher order equations with $m \geq 2$

REFERENCES

[1] B. Karpuz, R. N. Rath, S. K. Rath, On Oscillation and asymptotic behaviour of a higher order functional difference equation of neutral type, Int. J. Difference Equ. 4(1), 2009 , . 69–96.

[2] R. P. Agarwal, M. M. S. Manuel and E. Thandapani, Oscillatory and nonoscillatory behavior of second order neutral delay difference equations, Math. Comput. Modelling, . 24, (1), 1996, 5–11.

[3] R. P. Agarwal, S. R. Grace, Oscillation of higher order nonlinear difference equations of neutral type, Appl. Math. Lett., 12(8), 1999, 77–83.

[4] N. Parhi and A. K. Tripathy, Oscillation of forced nonlinear neutral delay difference equations of first order, Czech. Math. J., 53(128), no. 1, 2003, 83–101.

[5] N. Parhi and A. K. Tripathy, Oscillation of a class of nonlinear neutral difference equations of higher order, J. Math. Anal. Appl., . 284(2), 2003, 756–774.

[6] R. N. Rath, S. K. Rath and B. L. S. Barik, Non-oscillation criteria for the solutions of higher order functional difference equations of neutral type, International J. Difference equ. 3(2), 2008, 289-304.

[7] R. N. Rath, N. Misra and S. K. Rath, Sufficient conditions for oscillatory behaviour of a first order neutral difference equation with oscillating coefficients, Acta Mathematica Academiae Paedagogicae Nyiregyhaziensis (AMAPN), 25(1), 2009, 55-63 (www.emis.de/journals)

[8] P. Sundaram and V. Sadhasivam, On forced first order neutral difference equations with positive and negative coefficients, J. Indian Acad. Math., 26,(1), 2004, 211–227.
[9] E. Thandapani, P. Sundaram, J. R. Graef and P. W. Spikes, Asymptotic behaviour and oscillation of solutions of neutral delay difference equations of arbitrary order, Math. Slovaca, 47, 1997, 539–551.
[10] E. Thandapani, M. M. S. Manuel, J. R. Graef and P. W. Spikes, Oscillation of a higher order neutral difference equation with a forcing term, Int. J. Math. Math. Sci., 22(1), 1999, 147–154.
[11] E. Thandapani and K. Mahalingam, Necessary and sufficient conditions for oscillation of second order neutral difference quations, Tamkang J. Math., 34,(2) , 2003, . 137–145.
[12] M. K. Yıldız, ¨ O. ¨Ocalan, Oscillation results for higher order nonlinear neutral delay difference equations, Appl. Math. Lett., 20(3), 2007, . 243–247.
[13] Y. Zhou and Y. Q. Huang, Existence for nonoscillatory solutions of higher-order nonlinear neutral difference equations, J. Math. Anal. Appl., 280(1), 2003, . 63–76.
[14] T. H. Hilderbrandt, Introduction to the Theory of Integration, Academic Press, New-York, 1963.

BIOGRAPHIES

The first author Dr.Radhanath Rath, is Born on 27-03-1958 at ASKA(Odisha). He is currently working as HOD MATH/MCA at Khalikote autonomous college Berhampur. Before this he was working as Professor and Head of Math. Dept. at VSSUT, Burla (Formerly UCE BURLA). Dist. Sambalpur(Odisha). His area of interest in research is "Qualitative behavior of differential/Difference equations" and published nearly 40 articles in different international journals.The second and third author are working as Asst.Prof. in MATH and MCA respectively at Gandhi Eng.college,Bhubaneswar(Odisha).Their area of interest is "Qualitative behavior of Difference equations" and published some articles in different international journals.

CORRESPONDING AUTHOR: RADHANATH RATH, P. G., DEPARTMENT OF MATHEMATICS, KHALLIKOTE

AUTONOMOUS COLLEGE, BERHAMPUR, 760001 ORISSA, INDIA.

AJIT KUMAR BHUYAN, Department of Mathematics, G.E.C, Bhubaneswar, Odisha.

SUBHENDU KUMAR RATH, DEPARTMENT OF COMPUTER SCIENCE & ENGINEERING, G. E. C., BHUBANESWAR, ORISSA, INDIA.